Invisible mean field dynamos

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We provide examples of α^2 dynamos in spheres which generate magnetic fields that are confined to the conductor and are therefore undetectable in the surrounding vacuum.

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The generation of planetary and stellar magnetic fields is generally ascribed to the dynamo effect through which mechanical energy stored in the motion of a liquid conductor is converted into mechanical energy (for an overview of dynamo theory, see, e.g., [1]). The only probe available for investigation of the field generation mechanism is observation of the magnetic field in the insulating region surrounding the conductor. Basic theoretical models of the phenomenon solve the induction equation for the magnetic field assuming a prescribed velocity field in a homogeneous liquid conductor filling a sphere. Further simplification is achieved if the fluid flow occurs on typical length scales much smaller than the radius of the sphere. One then arrives at ''mean field electrodynamics'' [2]. In this paper we prove the existence of solutions of the mean field dynamo equation that are zero outside the region occupied by the conductor. The magnetic field is thus trapped inside the conductor and ''invisible'' for any exterior observer. A dynamo operating in a celestial body might therefore go completely unnoticed.

The absence of sizable fields on Mars and Venus is generally not interpreted in terms of invisible dynamos. It is rather thought that these planets are not dynamos at all, but Mars appears to have possessed a magnetic field in the past [3]. However, another planet may be of interest here. Saturn's magnetic field has dipole, quadrupole, and octupole contributions which are axisymmetric according to satellite measurements. No departure from axisymmetry has actually been detected. Yet no strictly axisymmetric dynamo field exists according to Cowling's theorem so that nonaxisymmetric components of the low multipoles are likely to be hidden in Saturn's conducting domain [4].

Invisible dynamos are one facet of the following broader (inverse) problem: The observer seeks to determine the motion of the liquid conductor from measurements of the magnetic field in the surrounding vacuum. This problem seems to be highly underdetermined since the exterior magnetic field enforces boundary values only for the poloidal magnetic field component in the conducting domain. Thus, the toroidal magnetic field component as well as the flow field have to be deduced from these data. Moreover, even if the magnetic field is completely known the flow field cannot be determined unambiguously since only the component perpendicular to the magnetic field enters the dynamo equation. A step toward the solution of the inverse problem has been undertaken by Lortz [5]: In the case of a given steady magnetic field whose field lines are constrained onto nested toroidal surfaces necessary conditions have been formulated for the existence of a well-defined flow field satisfying the steady induction equation. Unfortunately, these conditions are not useful from a practical point of view.

Despite the presumed indetermination of the inverse problem no example seems as yet to be known of two dynamos yielding identical vacuum fields. Here, we provide such an example by presenting dynamos that produce no exterior field and therefore cannot be distinguished by an external observer from each other or from a motionless core.

In the context of invisible dynamos one should note a theorem by Bondi and Gold predicting that motion of a perfectly conducting fluid in a simply connected domain like a sphere cannot generate an exterior field with finite dipole moment from an infinitesimal seed field [6]. This behavior has also been found within the approximations of mean field electrodynamics [7] and corresponds to the vanishing of the vacuum magnetic field if the magnetic Reynolds number of the motion inside the sphere tends to infinity, as observed in numerical simulations [8]. However, at any finite conductivity, there is a finite field strength in the vacuum and, therefore, these dynamos do not qualify as invisible dynamos.

The basic problem connected with invisible dynamos is the fact that they are solutions of an overdetermined boundary-value problem. The usual kinematic dynamo problem consists in finding a nondecaying (in time) solution of the induction equation with prescribed velocity field in a given domain S of liquid conductor, which matches an exterior vacuum field decreasing at least as fast as a dipole field at infinity. This is a well-posed problem in entire space. Requiring a vanishing vacuum field reduces the problem to a boundary-value problem in S, however, with more conditions on ∂S than in the standard case. The question arises for which velocity fields and for which domains the overdetermined boundary-value problem still has solutions. In the case of the induction equation this question is not easy to answer numerically because solutions are necessarily three dimensional and need a large number of components in a spectral representation. The signature of an invisible dynamo in this framework is a component vector being exactly zero. Since one can never exactly find a root numerically, it becomes tedious to demonstrate the mere existence of a root of a large system of equations. In one dimension, a sign reversal of a continuous function unambiguously identifies a root. In the application below, the presence of a zero of a system of two equations in two unknowns is proved by showing that the zero contour lines of the two functions cross. In this two-dimensional case, it is once again enough to consider only the signs of the functions.

In the framework of mean field electrodynamics the mean electromotive force is parametrized in the simplest case by a single scalar quantity $\alpha(\mathbf{x})$. The corresponding mean field equation leads in the case of invisible solutions likewise to an overdetermined boundary-value problem. Dynamo solutions in general are, however, much easier to find (Cowling's theorem, for example, does not apply). If, moreover, in a spherical domain α is assumed to depend only on the radial direction the problem even reduces to a one-dimensional problem. This situation is considered in the following. First, a special (unphysical) case is treated fully analytically. It serves to demonstrate in principle the solvability of the overdetermined problem and to display the kind of constraints the α field has to satisfy. The existence of invisible dynamos in the general case is then demonstrated on a numerical basis. Finally, critical magnetic Reynolds numbers for these dynamos are determined and compared to those of ordinary (visible) dynamos.

Invisible mean field dynamos in the unit sphere S with a pure α^2 mechanism are solutions of the boundary-value problem

$$\partial_t\mathbf{B}-\Delta\mathbf{B}+\nabla\times(\alpha\mathbf{B})=\mathbf{0},\quad \nabla\cdot\mathbf{B}=0 \quad \text{in } S, \tag{1a}$$

$$\mathbf{B}=\mathbf{0} \quad \text{on } \partial S. \tag{1b}$$

Using spherical polar coordinates (r,θ,ϕ), $\alpha=\alpha(r)$ is assumed to be a function of r only and the magnetic field $\mathbf{B}$ is decomposed into its poloidal and toroidal parts and expanded in spherical harmonics:

$$\mathbf{B}(\mathbf{x},t)=\nabla\times\nabla\times\left(\sum_{l,m} p_l^m(r,t)P_l^m(\cos\theta)e^{im\phi}\hat{\mathbf{r}}\right) +\nabla\times\left(\sum_{l,m} t_l^m(r,t)P_l^m(\cos\theta)e^{im\phi}\hat{\mathbf{r}}\right) \tag{2}$$

with $P_l^m(\cos\theta)$ denoting the associated Legendre functions and $\hat{\mathbf{r}}$ the unit vector in the radial direction. With the representation (2) for $\mathbf{B}$ Eq. (1a) becomes equivalent to the system of equations for the components $p_l^m(r,t)$ and $t_l^m(r,t)$:

$$\begin{gathered}\partial_t p_l^m - D_l p_l^m + \alpha t_l^m = 0,\\ \partial_t t_l^m - D_l t_l^m - \partial_r\alpha\partial_r p_l^m - \alpha D_l p_l^m = 0\end{gathered} \tag{3}$$

with $D_l := \partial_r^2 - l(l+1)/r^2$. The boundary conditions read

$$p_l^m = t_l^m = 0 \quad \text{at } r=0, \tag{4a}$$

$$p_l^m = \partial_r p_l^m = t_l^m = 0 \quad \text{at } r=1. \tag{4b}$$

The conditions (4a) ensure a well-defined magnetic field at the origin whereas (4b) derives from the boundary condition (1b). Equations (3) do not depend on m; thus the superscript is henceforth omitted.

Considering the boundary conditions a prime candidate for an invisible dynamo would be a purely toroidal field. This possibility is obviously ruled out by Eqs. (3). In fact, purely toroidal dynamos are possible for neither the mean field equation with general scalar α field nor the induction equation [9].

The stationary monopole case ($\partial_t\equiv 0$, $l=0$) is amenable to analytical treatment. Eliminating the toroidal field and introducing the variable $x:=1-r$ as well as the fields $P(x) := p_0(r)$, $\tilde{\alpha}(x):=\alpha(r)$ one obtains from Eqs. (3) and (4)

$$\left(\frac{1}{\tilde{\alpha}}P''\right)''+(\tilde{\alpha}P')'=0 \tag{5}$$

(the prime means d/dx) with the boundary conditions $P(0)=P'(0)=P''(0)=P(1)=P''(1)=0$. In deriving the boundary conditions $\tilde{\alpha}(0)\neq 0$, $\tilde{\alpha}(1)\neq 0$ has been assumed. Further simplification is achieved by introducing the variable $Q(x)$ by $P(x)=\int_0^x Q(\tilde{x})d\tilde{x}$ and integrating Eq. (5) once:

$$\left(\frac{1}{\tilde{\alpha}}Q'\right)'+\tilde{\alpha}Q=C. \tag{6}$$

The boundary conditions transform into $Q(0)=Q'(0) =Q'(1)=0$ and the mean value condition $\int_0^1 Q(x)dx=0$. The complete solution of Eq. (6) reads

$$Q(x)=A\sin y(x)+B\cos y(x)+C[\sin y(x)\mathrm{Siny}(x) +\cos y(x)\mathrm{Cosy}(x)] \tag{7}$$

with $y(x):=\int_0^x\tilde{\alpha}(\tilde{x})d\tilde{x}$, $\mathrm{Siny}(x):=\int_0^x\cos y(\tilde{x})d\tilde{x}$, $\mathrm{Cosy}(x) :=-\int_0^x\sin y(\tilde{x})d\tilde{x}$, and arbitrary constants A and B. The boundary conditions $Q(0)=Q'(0)=0$ are satisfied for $A =B=0$, whereas the remaining conditions impose the following integral conditions on $y(x)$ or $\alpha(x)$, respectively:

$$\sin y(1)\mathrm{Cosy}(1)-\cos y(1)\mathrm{Siny}(1)=0, \tag{8}$$

$$\int_0^1[\sin y(x)\mathrm{Siny}(x)+\cos y(x)\mathrm{Cosy}(x)]dx=0. \tag{9}$$

Equations (8) and (9) are in fact satisfied for a large class of functions $y(x)$. Assume, for example, $y(x)$ to be symmetric with respect to $x=1/2$, $y(x)=y(1-x)$; then $\sin y(x)$ and $\cos y(x)$ are symmetric and the functions $\overline{\mathrm{Siny}}(x):=\mathrm{Siny}(x) -1/2\mathrm{Siny}(1)$ and $\overline{\mathrm{Cosy}}(x)=\mathrm{Cosy}(x)-1/2\mathrm{Cosy}(1)$ are antisymmetric. Rewriting condition (9) in the form

$$\int_0^1[\sin y(x)\overline{\mathrm{Siny}}(x)+\cos y(x)\overline{\mathrm{Cosy}}(x)]dx=0$$

shows that this condition is satisfied. Because of $y(0) =y(1)=0$ condition (8) reduces now to $\mathrm{Siny}(1)=0$, which is satisfied, for example, if $y(x)$ has in addition the symmetry $y(1/2-x)+y(x)=(2n-1)\pi$ on the interval (0, 1/2) with n being an arbitrary positive integer. The function $y(x)=2\pi x$ on (0, 1/2) and $2\pi(1-x)$ on (1/2, 1) obviously has both symmetries (for $n=1$). It corresponds to the function $\alpha(r)$ shown in Fig. 1. Smooth functions $\alpha(r)$ are of course also admissible.

Numerical treatment is necessary in order to find solutions for arbitrary l. For α we choose two simple profile functions

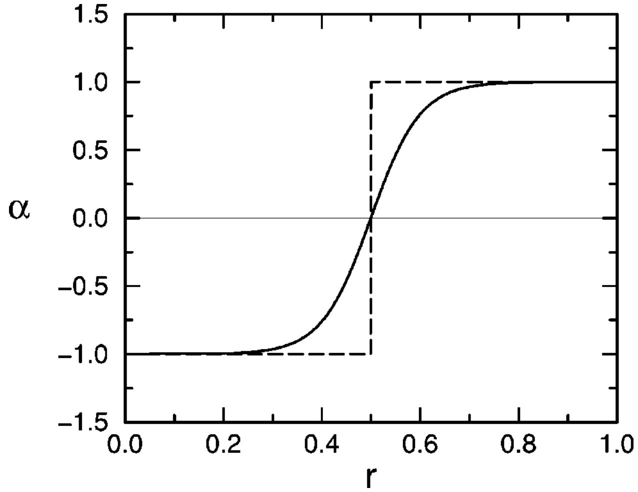

FIG. 1. Profiles of $\alpha(r)$: The (normalized) step function found in the analytical treatment (dashed) and a continuous approximation to the step function used in Figs. 2 and 3 below (solid line, profile α_I with $\delta=10$, $\alpha_0=1$, and $r_0=0.5$).

parametrized by α_0, the strength of the α effect, and an additional parameter (r_0 or c_0) determining the shape. The first profile $\alpha_I=\alpha_0\tanh[-\delta(1-r-r_0)]$ imitates (in the limit $\delta\to\infty$) the step function used for $l=0$; the second one $\alpha_{II}=\alpha_0[(1-c_0)\sin\pi r-c_0\sin 2\pi r]$ varies for $0\leqslant c_0\leqslant 1$ between a positive function and one with a sign reversal.

In order to find invisible dynamos we focus on time independent solutions with $p_l(r=1)=p_l'(r=1)=t_l(r=1)=0$, which ensures confinement of the magnetic field. Since the problem is linear in p_l and t_l, one can arbitrarily choose the normalization $t_l'(r=1)=1$. These four conditions serve as initial conditions for a fourth order Runge-Kutta integration that progresses p_l and t_l from $r=1$ to $r=0$. In the case of profile α_I, the boundary conditions at $r=0$ are satisfied only for special choices of r_0 and α_0. $p_l(r=0)$ and $t_l(r=0)$ are the two functions of α_0 and r_0 whose roots are being sought. Crossings of the zero contour lines of $p_l(r=0)$ and $t_l(r=0)$ in the (α_0,r_0) plane correspond to stationary solutions of the mean field equation with zero exterior field.

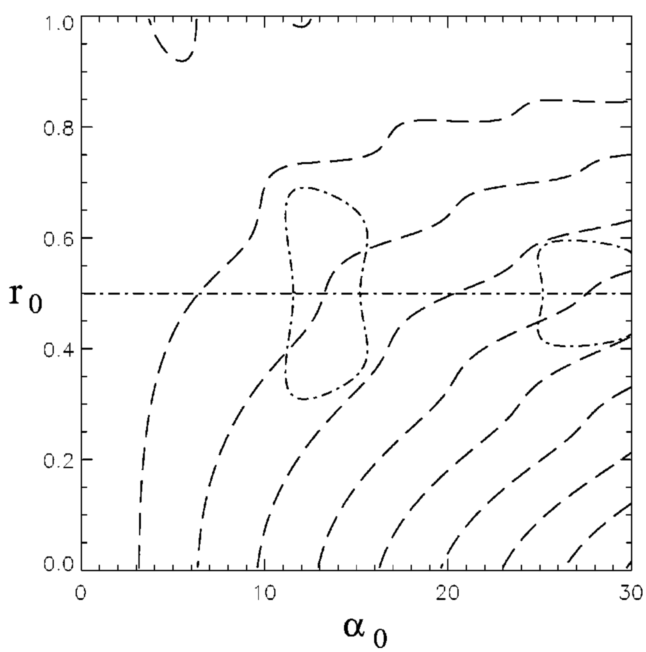

FIG. 2. The dot-dashed and dashed curves correspond to zeros of $p_0(r=0)$ and $t_0(r=0)$ in the (α_0,r_0) plane, respectively, obtained with the integration procedure described in the text. α is given by $\alpha=\alpha_0\tanh[-10(1-r-r_0)]$.

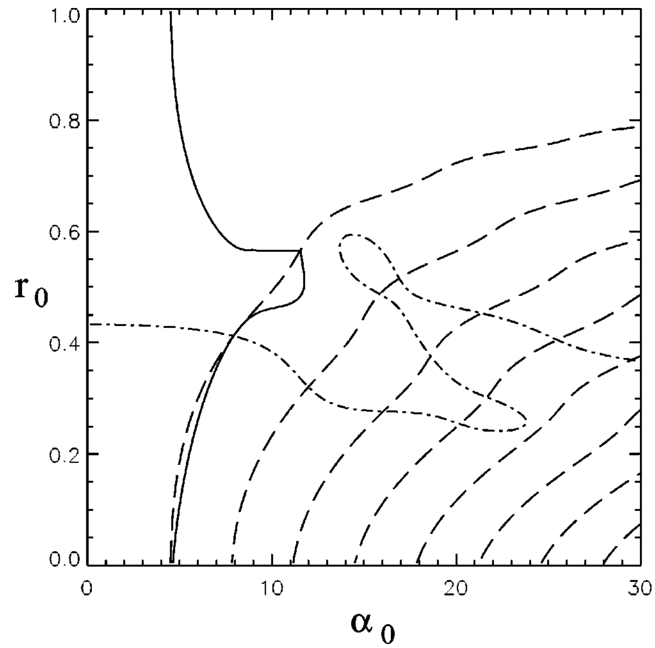

FIG. 3. Stability boundary (solid line) of α^2 dynamos with α given by $\alpha_0\tanh[-10(1-r-r_0)]$. The dot-dashed and dashed curves correspond to zeros of $p_l(r=0)$ and $t_l(r=0)$, respectively, for $l=1$.

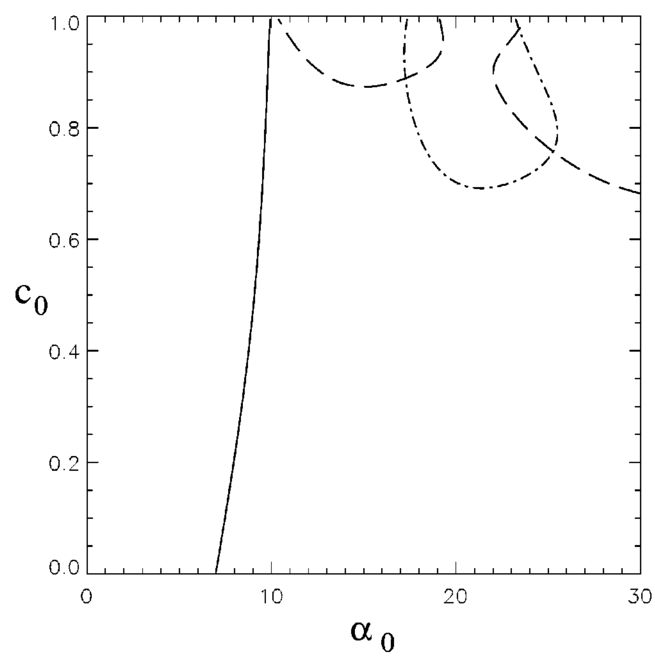

FIG. 4. Same as Fig. 3 for $\alpha=\alpha_0[(1-c_0)\sin\pi r-c_0\sin 2\pi r]$ and $l=2$.

Figure 2 shows the case $l=0$ considered analytically above for the profile α_I with $\delta=10$. A solution of the overdetermined problem exists in the vicinity of $r_0=0.5$, $\alpha_0=2\pi$. The agreement with the solution found analytically improves if δ is increased. Note that the other solutions contained in Fig. 2 correspond to further symmetric solutions as well as to nonsymmetric ones.

Figure 3 repeats the same study for the physically more relevant case $l=1$. The most interesting solution is the one at $l=1$, $r_0=0.412$, $\alpha_0=7.92$, which will be shown below to lie on the stability boundary.

The profile α_{II} is investigated in Fig. 4, this time for $l=2$. The results confirm the intuitive expectations nourished above. In order to obtain an invisible dynamo solution, comparable volumes of the sphere must be occupied by α effect of each sign. Accordingly, the profile α_I yields invisible solutions for values of r_0 in a band around 0.5, and α_{II} requires a c_0 larger than roughly 0.6 for confined dynamo modes to exist.

For any value of r_0 or c_0 there is a critical value of α_0 at which dynamo action starts. From the observational point of view, solutions of the induction equation of the invisible type are of interest only if they lie on the stability boundary, i.e., if there is no other mode that grows as time goes on at the same r_0 or c_0. Only in this case do all magnetic modes decay except the one that is hidden inside the conductor. For the profile α_I, it has been checked that the onset for dynamo action occurs for $l=1$ with a nonoscillatory magnetic field. The corresponding stability boundary obtained with a standard shooting method is also shown in Fig. 3. A mode crossing occurs for r_0 near 0.55, which explains the segment of horizontal line in the stability curve. For this profile, a value of r_0 indeed exists for which dynamo action starts with an invisible field. The same holds true at nearby parameters. For instance, for $\alpha=\alpha_0\tanh\{-2[(1-r)-r_0]\}$ an invisible dynamo occurs at $l=1$, $r_0=0.426$, and $\alpha_0=15.18$. Profile α_{II} on the other hand always leads to onset with a visible magnetic field. The stability boundary for $l=2$ is again included in Fig. 4. For $c_0<0.6$, fields with $l=1$ are actually preferred at onset but they all penetrate the vacuum region.

In summary, it has been demonstrated that there are solutions of the mean field dynamo equation in a sphere that are zero in the vacuum region. In particular, there are solutions that are invisible at the onset of dynamo action. The question arises, of course, whether invisible solutions of the induction equation also exist. A positive answer can be expected since the mean field equation derives from the induction equation under the assumption of scale separation in the velocity field and—from a mathematical point of view—the overdetermined character of the boundary-value problem is the same for both equations. A direct translation from a mean field dynamo to a real dynamo would consist in choosing a velocity field that approximately reproduces the α effect used above. However, a small scale magnetic field would also result which in general is visible from the outside. On the other hand, that field can be made arbitrarily small in comparison with the main field. Suppose that eddies of typical size l_0 and typical velocity u_0 exist in a fluid of diffusivity λ. The ratio of small scale field $\mathbf{b}$ to large scale field $\mathbf{B}$ is then given in order of magnitude by $|\mathbf{b}|/|\mathbf{B}|\sim u_0 l_0/\lambda$, whereas $\alpha\sim u_0^2 l_0/\lambda$ [1]. By fixing α and choosing a small l_0 one can obtain arbitrarily small ratios $|\mathbf{b}|/|\mathbf{B}|$. A small l_0 is of course inpractical for numerical computations.

In addition, it is difficult to convincingly prove by numerical means the existence of a strictly invisible dynamo in spherical geometry due to the large number of equations that need to be satisfied simultaneously (as was already mentioned in the Introduction). The investigation of cylindrical dynamos with simple flow field is therefore more appropriate. Helical flows of Ponomarenko type [10], in which field amplification is concentrated on some isolated surfaces, have been examined but without success. More general flows that allow for a continuous radial variation of the flow field are currently under study.

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